A simple theory of capillary-gravity wave turbulence

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ABSTRACT

Employing a recently proposed "multi-wave interaction" theory [JFM, 243, 623-625], spectra of capillary-gravity waves are derived, This case is characterized by a rather high degree of nonlinearity and a complicated dispersion law. The resultant absence of scale invariance makes this and some other problems of wave turbulence (e.g., nonlinear Rossby waves) intractable by small-perturbation techniques, even in the weak-turbulence limit. The analytical solution obtained in the present work is shown to be in good agreement with experimental data, Its low- and high-frequency limits yield power-laws characterizing spectra of purely gravity and capillary waves, respectively. In the limits of weak and strong nonlinearity, these reduce to the Zakharov-Filonenko and Phillips spectra, respectivel y.

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I. Introduction

The subject of this paper is turbulence of surface gravity-capillary waves, although the formalism can be applied to other problems of nonlinear wave dynamics, such as internal waves, Rossby waves, acoustic turbulence, etc. The weak turbulence theory (W'l-l') presently available for these problems [Zakharov, L'vov and Falkovich, 1992] proved successful in many cases. However, some of its constraints greatly limit its scope. Duc to formidable mathematical difficulties, WTT cannot account for higher-order nonlinear effects. Besides, WTT requires scale invariance (as yielded by a power-law type of dispersion law) and Idealization of external sources and sinks - m yield practical results even for weakly nonlinear problems. Therefore, some intuitive and less formal approaches may prove more useful in many cases. An example is given by Phillips [1985] where weak turbulence of surface gravity waves is considered with the source functions continuously distributed in the wavenumber space. A similar approach, but going beyond the weak turbulence limit (and called the "multiwave interaction theory," or MIT for short) was suggested recently [Glazman, 1992] to explain observed variations in the exponent for power law spectra of surface gravity waves. The Kolmogorov assumption of locality of nonlinear wave-wave interactions is crucial in these theories. Provided this assumption remains approximately valid for an increased number of the resonantly Fourier components, MIT could in principle be applied to a broad class of problems. Indeed, it does not require the lowest degree of nonlinearity, simple dispersion laws or simple expressions for the wave energy density, and it can be used for weakly nonconservative systems - as demonstrated earlier. However, due to its heuristic nature, MIT requires thorough experimental verification.

The case of capillary-gravity waves is characterized by a highly complex expression for the potential energy,

$$u== \tfrac{1}{2}\rho g\int \eta^2 d\mathbf{x} + \sigma\rho\int (\sqrt{1+|\nabla\eta|^2}-1)\, d\mathbf{x} \qquad (1.1)$$

where $\eta=\eta(x,t)$ is the elevation of the fluid surface above the zero-mean level, g is the acceleration of gravity, and σ is the coefficient of surface tension, 7', divided by fluid density ρ. The dispersion law is

$$\omega^2 = gk + \sigma k^3 \;, \qquad (1.2)$$

which, while permitting three-wave resonant interactions, eliminates scale-invarifincc (actually, self-affinity) of the wave field. Characteristic wavelengths at which (1. 1) and

(1 .2) are relevant extend from tenths and up to tenor more centimeters - the waves amenable to accurate elaborator investigation. Thus, nlike other examples of wave turbulence, the capillary -gravity waves are highly interesting as a test case. Besides, these waves are primarily responsible for radar backscatter by the ocean surface, and thus are of great practical interest.

In section 2, the theoretical approach is briefly reviewed emphasizing a few important points that escaped the author's attention in the earlier paper [Glazman, 1992]. Spectra of capillary-gravity waves are derived in section 3, and experimental comparison is presented in section 4.

2. Multiwave-interaction theory for surface gravity-capillary waves

Let us consider a conservative spectra] flux of wave energy. The external energy source acts at lower frequencies - outside our inertial subrange. Therefore, a specific mechanism of wave generation is not addressed here. The rate Q of energy input, assumed to be known, equals the rate of energy transfer down the spectrum. Following the earlier reasoning [Glazman, 1992], Q is related to the characteristic t ime of nonlinear wave-wave interaction (the "turnover time"), t_n, and the characteristic energy E_n transferred from a cascade step n to step $(n+1)$ by

$$\rho Q = E_n / t_n \tag{2.1}$$

where the water density ρ appears because Q is taken per unit mass of water. Provided E_n and t_n can be expressed in terms of k, ω and wave amplitude a, equation (2. 1) allows one to derive the spectrum by means of clement ary algebra (e.g., [Frisch et al., 1978]). Let us express these parameters in terms of the relevant quantities.

An approximate equi-partition of energy between the kinetic and the potential parts allows one to write the surface density of the total wave energy, E, as

$$E \approx \rho\left[g\left\langle\eta^2\right\rangle + \sigma\left\langle(\nabla\eta)^2\right\rangle\right] \tag{2.2}$$

where the angular brackets denote ensemble average. To pass from (1.1) to (2.2) wc assumed $\langle(\nabla\eta)^2\rangle << 1$ which is reasonably well justified for natural seas (e.g., [Cox and Munk, 1954]). Validity of this assumption for a laboratory environment will be discussed in section 4. The energy E is related to the spectral density of the wave energy by:

$$E = \int S(\omega)d\omega = \iint F(k,\theta)kd\theta dk\,, \tag{2.3}$$

where the integration is carried out over all wavcnLiinbers/frequencies. Here, $S(\omega)$ is the frequency spectrum and $F(k,\theta)$ is the two-dimensional wavenumber spectrum of the wave energy.

The amount of energy, E_n, transferred by the cascade mechanism is estimated as:

$$E_n = \int_{\omega_n}^{\omega_{n+1}} S(\omega)d\omega \tag{2.4}$$

where (ω_n, ω_{n+1}) is the width of a cascade step (which must be much smaller than the width of the inertial range), and the ratio $r = \omega_{n+1} / \omega_n$ is constant and sufficiently greater than unity - as required by the assumption of locality of wave-wave interactions in the frequency space. Indeed, differentiating (2,4) over ω_n yields $-dE_n / d\omega_n = S(\omega_n) - rS(r\omega_n) = S(\omega_n)[1 - r^{1-p}]$, where the latter equality is valid for wave spectra of type $S(\omega) \propto \omega^{-p}$. Therefore, provided the spectrum rolls off sufficiently fast (i.e., $r^{1-p} << 1$), we have:

$$S(\omega) \approx -dE(\omega) / d\omega \tag{2.5}$$

Although the spectrum being derived here has a more complicated shape than that given by ω^{-p}, the above approximation can be easily checked *a posteriori*. From (2.2) it is obvious that E_n for gravity-capillary waves can be written as

$$E_n \approx \rho[ga_n^2 + \sigma(a_n k_n)^2] \tag{2.6}$$

Here, a_n is the Fourier amplitude of surface oscillation at the frequency/wavenumber scales ω_n and k_n, corresponding to the n-th step in the spectral cascade.

The derivation of the turnover time is formally based on the scaling of the collision integral in the kinetic equation [Zakharov and L'vov, 1975; Larraza et al, 1990] for the wave action spectral transfer. It is also useful to introduce this timescale in a more general, although less formal fashion. To this end we notice that the nonlinearity of wave processes is measured by the ratio, ε, of the fluid particle velocity, u, to the wave phase velocity, ω/k [Witham, 1974]. Since fluid particles in a surface wave execute an approximately orbital motion in the vertical plane with the radius equal to the wave amplitude and the period $2\pi/\omega$, the value of u at a given scale is estimated as $a_n\omega_n$. Respectively, the ratio $u/(c/\omega)$ is

$$\varepsilon_n \equiv \frac{u_n}{\omega_n / k_n} \approx \frac{a_n\omega_n}{\omega_n / k_n} = a_n k_n \tag{2.7}$$

This quantity represents the small parameter in deterministic perturbation theories. However, since the kinetic equation for the wave action, $N(k) = F(k)/\omega$, (or wave energy, $F(k)$) spectral transfer is derived for second statistical moments of the fields, the equations of statistical theory are developed in powers of ε^2. Terms (i.e., collision integrals) of order ε^2 correspond to three-wave interactions, while each additional Fourier component accounted for in the interaction integral adds new terms which are ε^2 times as great as a preceding term. The v-th term is of order $\varepsilon^{2(v-2)}$. Respectively, the characteristic time of nonlinear wave-wave interactions increases as the number of interacting harmonics grows.

For 3-wave interactions, this time is given by $t^{-1} \approx \omega\varepsilon^2$, and for an arbitrary number, v, we have [Larraza et al., 1990]:

$$t^{-1} \approx \omega\varepsilon^{2(\nu-2)} \tag{2.8}$$

Formally, the kinetic equation is written as

$$\partial N / \partial t + \nabla_k \bullet \mathbf{T(k)} = \mathbf{p(k)} \ , \tag{2.9}$$

where ***p(k) is*** the spectral density of the input flux of wave action (from wind), and $\nabla_k \bullet$ **7'(k)** denotes the spectral density of the action flux due to all wave-wave interactions to order ***v:***

$$\nabla_k \bullet \mathbf{7'(k)} = I_3 + I_4 + \ldots + I_\nu \tag{2.10}$$

I_ν are collision integrals accounting for interactions among v waves satisfying resonance conditions

$$\begin{aligned} \omega_0 \pm \omega_1 \pm \ldots \pm \omega_\nu &= 0 \\ \mathbf{k}_0 \pm \mathbf{k}_1 \pm \ldots \pm \mathbf{k}_\nu &= 0 \end{aligned} \tag{2.11}$$

(non-resonant terms can be eliminated by appropriate canonical transformations [Zakharov et al., 1992]). For gravity waves, the minimum number of resonantly interacting components is 4, while for capillary waves it is 3.

It has been argued earlier [Glazman, 1992] that intermittently occurring, rare events of steep and breaking waves (characterized by a locally high nonlinearity, hence a large, or even infinite, number of interacting Fourier components forming individual highly non-linear wavelets), result in an increased mean (over a large time interval and large surface area) value of v. While this v may be substantially greater than the minimum resonant number appearing in WI"]', the energy and action transfer may still be dominated by the weakly nonlinear inertial cascade. Thus, the "effective" v is introduced as an unknown function of the problem, the assumption of locality of wave-wave interactions in the wavenumber space remaining in force, Let us notice that the turnover time given by (2.8) for the highest-order term in (2,10) is the slowest of all the times for "partial" fluxes in (2.10). Therefore, although the total flux of the wave action, $\nabla_k \bullet$ **7'(k),** is comprised of many partial fluxes I_3, 1_4, etc., the appropriate characteristic time scale for the integral transfer is given by (2.8),

We consider the case when the external input is concentrated at wavenumbers below certain k_0 marking the high-wavenumber boundary of the "generation range. " Therefore, at $k > k_0 : p(k) = O$, and the spectral flux is purely inertial. It is given by

$$\rho Q = \int_0^{k_0} \omega(k) k dk \int_{-\pi}^{\pi} p(k,\theta) d\theta \tag{2.12}$$

Respectively, equation (2.9) for the inertial range yields

$$E_n \omega_n (a_n k_n)^{2(V-2)} \approx \rho Q \ (= const) \tag{2.13}$$

where $n \geq 1$.

Using (1.2) and (2.6), equation (2.13) results in

$$E_n \approx \rho Q^{1/(\nu-1)} \sigma^{(\nu-2)/(\nu-1)} \omega_n^{-1/(\nu-1)} \Phi_\nu(\omega_n). \tag{2.14}$$

where $\Phi_\nu(\omega_n) = \left[\frac{1+M(\omega_n)}{M(\omega_n)}\right]^{(\nu-2)/(\nu-1)}$, $M(\omega) = [k(\omega)/\kappa]^2$ (2.15)

and $1/\kappa = (\sigma/g)^{1/2}$ gives the characteristic lengthscale of the problem (where the phase speed is at minimum) The explicit dependence of k on ω , as follows from (1 .2), is

$$k(\omega) = u_1(\omega) + u_2(\omega) \, , \tag{2.16}$$

where

$$u_{1,2}(\omega) = \sqrt[3]{\frac{\omega^2}{2\sigma} \pm \sqrt{D(\omega)}} \qquad \text{and} \qquad D(\omega) = \frac{4g^3 + 27\omega^4\sigma}{108\sigma^3} \tag{2.17}$$

Based on (2.14) and (2.5), the energy spectrum is found as

$$S(\omega) = \alpha\rho \left(\frac{Q}{\sigma}\right)^{1/(\nu-1)} \frac{\Phi_\nu(\omega)}{\sigma} \left[1 + \frac{4(\nu-2)}{1+3M(\omega)}\right] \omega^{-\nu/(\nu-1)} \tag{2.18}$$

where α is a ("Kolmogorov") constant of proportionality. "l'he short-wave limit of (2. 18) is obtained by setting $M(\omega) \to \infty$, hence $\Phi_\nu(\omega) \to 1$. In a special case of $\nu=3$, this yields the Zakharov-Filonenko spectrum [Zakharov and Filonenko, 1967] of weakly nonlinear capillary waves. It is also easy to check that for v=4, the long-wave limit of (2. 18) yields the Zakharov-Filonenko spectrum [Zakharov and Filonenko, 1966] of gravity waves.

3. Wave spectra

Relationships between the energy spectrum (2. 18) and the spectra of surface height and surface gradient (i.e., wave slope) arc more complicated than those for purel y gravity and purely capillary waves. Specifically, as follows from (2.2), the spectrum Of surface height variation is related to (2.18) by

$$S_\eta(\omega) = \frac{S(\omega)}{\rho g[1+M(\omega)]} \tag{3.1}$$

It is easy to verify that (3.1) reduces to the well-known Zakharov-Filonenko and Phillips spectra when the appropriate limits are t aken. in the wavenumber domain, the two-dimensional spectrum of surface height variation (omitting the angular spread factor, $\Psi(\theta,k)$) is found as:

$$F_\eta(k) = k^{-1}\left[S_\eta(\omega)\frac{d\omega}{dk}\right]_{\omega=\omega(k)} \tag{3.2}$$

For simplicity, wc assume the following normalization condition for Y'(O, k):

$$\int_{-\pi}^{\pi} \Psi(\theta,k)d\theta = 1 \tag{3.3}$$

The two-dimensional spectrum of the wave slope modulus (again, the angular spread factor $\Psi(\theta,k)$ *is omitted) is* found as

$$F_{\nabla\eta}(k) = k^2 F_\eta(k) = \frac{\sqrt{kg}[1+3M(k)]}{2\rho g[1+M(k)]^{3/2}} S(\omega(k)) \tag{3.4}$$

It is useful to present these results in a nondimensional form by scaling all variables as follows:

$$k = K\kappa,\ \omega = \Omega\varpi,\ \tilde{Q} = \frac{Q}{(\varpi/\kappa)^3},\ \tilde{S}(\Omega) = \frac{S(\omega)\kappa^3}{\alpha\rho\varpi}\ , \tag{3.5}$$

where $\varpi = (g^3/\sigma)^{1/4}$ and $\kappa = (g/\sigma)^{1/2}$. In terms of K and Ω , the dispersion law (1 .2) takes the form

$$\Omega^2 = K + K^3 \tag{3.6}$$

The non-dimensional spectrum of wave energy becomes:

$$\tilde{S}(\Omega) = \tilde{Q}^{1/(\nu-1)} \frac{\Phi_\nu(\Omega)}{\nu-1}\left[1+\frac{4(\nu-2)}{1+3K^2(\Omega)}\right]\Omega^{-\nu/(\nu-1)} \tag{3.7}$$

and the non-dimensional spectrum of wave slope is:

$$\tilde{F}_{\nabla\eta}(K) = \frac{\sqrt{K}(1+3K^2)}{2(1+K^2)^{3/2}} \tilde{S}(\Omega(K)) \tag{3.8}$$

where $\tilde{F}_{\nabla\eta}(K) = \dfrac{K^2}{\alpha} F_{\nabla\eta}(k)$

4. Comparison with laboratory observations

To compare these results with the laboratory measurements by Jähne and Riemer [1990] (conducted in a large wave tank -100 m length), we need the "saturation function" $B(k) = k^2 F_{\nabla\eta}(k)$. An example of the J&R measurements is reproduced in Fig. 1. in the non-dimensional form, this is $\tilde{B}(K) = K^2 \tilde{F}_{\nabla\eta}(K)$ where $\tilde{B}(K) = B(k)/\alpha$. *The* values of $\tilde{Q}$ can be expressed via external parameters. Specifically, the energy flux is given by

$$Q = (\rho_a/\rho_w)C_q U^3 \tag{4.1}$$

where the density ratio is of order 10^{-3} and the integral transfer coefficient of the wave energy, C_q, is somewhere between 0.02 and 0.05 [Phillips, 1985; Glazman, 1993]. For the range of wind speed values tested in the J&R experiment, the non-dimensional energy flux varies between 0.5 and 30.

In principle, v can be related to the energy flux Q and the magnitude of the wave spectrum $F(k_0)$ at the low-frequency boundary k_0 of the given inertial subrange [Glazman, 1992], which would require matching (2.18) to a known spectrum of gravity waves. in other words, a "rigorous" determination of v require.s consideration of the whole wave-generation problem - a grand task that would take us wc]] beyond the scope of the present work. Besides, a rigorous determination of v might actually be irrelevant with respect to a wave tank situation. indeed, laboratory experiments greatly limit the wave age by inhibiting the development of the inertial cascade in the gravity range and thus creating in that range a highly artificial physical situation, Therefore, wc shall pick several values of v and $\tilde{Q}$ which appear reasonable. In particular, v should be an increasing function of wind (hence, of Q), Moreover, this function must display a saturation effect, i.e., its growth with an increasing $\tilde{Q}$ should slow down at high values of the latter.

According to (4.1) and Fig. 1, values $\tilde{Q}$ = 0.5, 1.0, 5,0 and 20 arc in the range of wind speeds tested by J&R . The theoretical predictions of $\tilde{B}(K)$ are illustrated in Fig. 2. Evidently, our curves arc in reasonable agreement with the measurements. "l'he comparison also yields an estimate of the Kolmogorov constant: $\alpha \approx 0.01$. The set of v selected in Fig. 2 can be plotted against Q, Fig. 3, or against wind speed, Fig. 4- using (4.1). It is also useful to derive an empirical fit based on Fig. 4. The result is:

$$\nu \approx 10\left(1-\frac{9}{U^2}\right) \tag{4.2}$$

where U is in m/sec.

The main quantitative discrepancy between the theoretically predicted spectrum and the measured spectrum is that the measured spectrum starts falling off at wavenumbers roughly twice those predicted in Fig. 2. This might be attributed in part to the fact that the surface tension coefficient characterizing a wave tank situation can hardly be as high as that for pure water (70 dyn/cm). If wc reduce σ by half, the quantitative agreement with the measured data will be much better. However, a more important reason for the disagreement is that we have used an assumption of a small wave slope to derive (2.2) from (1. 1), This assumption is likely to be violated in a wave tank experiment, and higher-order terms in the expansion of the square root in (1. 1) may have to be taken into account. In principle, the present heuristic approach makes such a refinement possible. However, with respect to sea waves, it is hardly justified.

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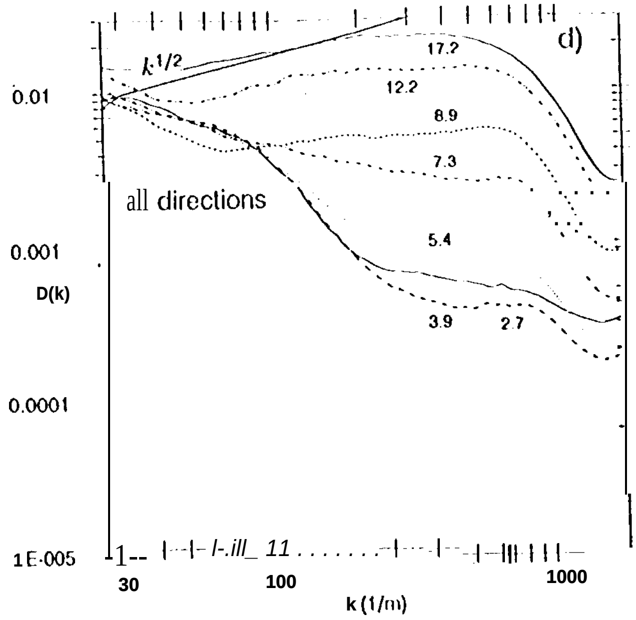
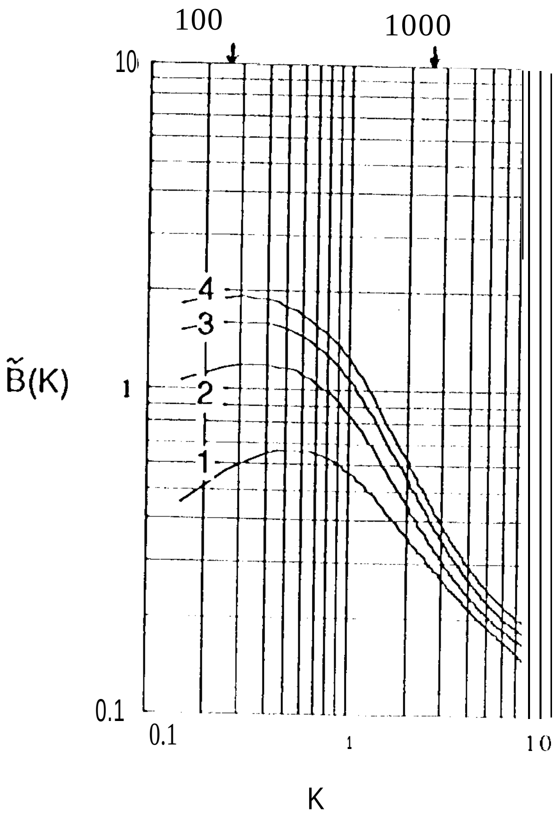


Fig. 1. The saturation function B (k) reported by Jahne & Riemer (1990) for wind speed values designated at the curves in m/sec.



Numbers at the curves designate:

1 : Q=0.5, ν=4.
2: Q=1.0, ν=7
3 : Q=5.0, ν=9
4: Q=20.0, ν=10

Figure 2. The saturation function $\tilde{B}(K)$ related to the wavenumber spectrum of wave slope, $\tilde{F}_\gamma(K)$, by $\tilde{B}(K) = \tilde{F}_\gamma(K)K^2$. Numbers (100 and 1000) on top give values of k in rad/m for the case of σ=70 cm^3/sec^2 and g=981 cm/sec^2

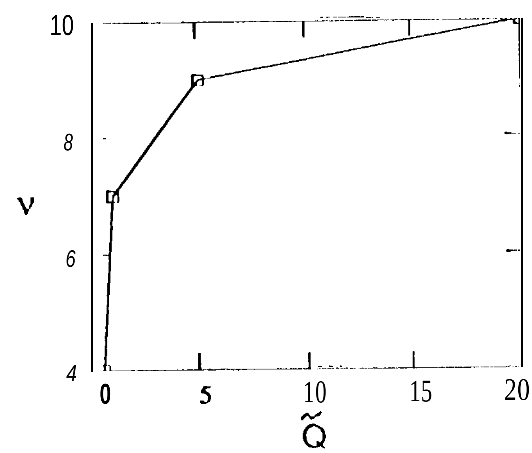


Fig.3. $\tilde{Q}$ and ν.

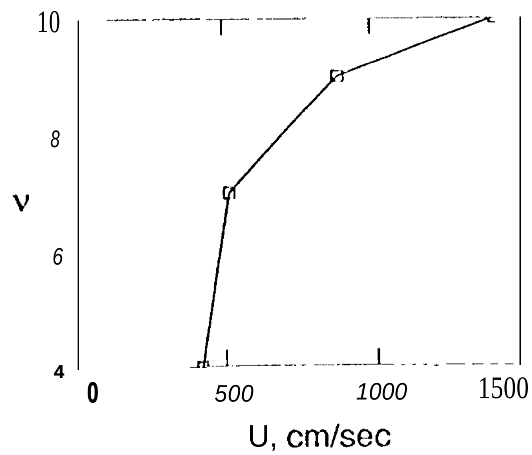


Fig.4. Wind speed corresponding to the values of $\tilde{Q}$ from Fig.3 and derived based on (4.1) with C_q=0.03.